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Direct Gauging of the Poincaré Group. III. Interactions with Internal Symmetries

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The problem of gauging matter fields with a Poincaré invariant action functional that admits an r parameter, semisimple group $G(r)$ of internal symmetries is considered. A minimal replacement operator for the total group $P_{10} \times G(r)$ is obtained, together with the requisite compensating 1-forms for both the Poincaré and the $G(r)$ sectors. A basis for $P_{10} \times G(r)$-invariant Lagrangian densities for the free fields is obtained. Restriction to Lagrangian densities that are at most quadratic in the associated curvature and torsion fields eliminates active coupling between the P_{10} free field Lagrangian and the $G(r)$ free field Lagrangian, although there is passive coupling that arises through the requirement of tensorial covariance under general coordinate transformations generated by the local action of the translation group. A superposition principle, modulo passive coupling, thus holds for quadratic free field Lagrangian for the total group: $L_{TOT} = L_P + L_{G(r)}$. Field equations for the matter fields and the compensating fields of the $G(r)$ sector are obtained. Both share the passive coupling to P_{10} that is required in order to achieve "tensorial" covariance, but only the matter fields couple directly to the Poincaré fields and only to the Lorentz sector. For "weak" Poincaré fields, the field equations for the matter fields and the compensating fields of the internal symmetries go over into the standard field equations of gauge theory for an internal symmetry group.

1. INTRODUCTION

A direct gauge theory for the Poincaré group was obtained in I (Edelen, 1985a) by realization of the Poincaré group as a matrix Lie group of automorphisms of an affine plane in V_5. The minimal replacement operator for the local action of the Poincaré group was obtained by the standard Yang–Mills construct. Its application to the standard frame bundle of Minkowski space gave the distortion 1-forms that constitute a frame bundle

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for a space U_4 with both curvature and affine Cartan torsion. The standard minimal coupling construct was then shown to lead to a system of field equations for the matter fields and the compensating 1-forms for the local action of P_{10}, in which the orbital and spin contributions of the gauge momentum energy of the matter fields were decoupled.

Further studies reported in II (Edelen, 1985b) analyzed the implications of P_{10} invariance of the total Lagrangian density. This led to explicit calculations of a basis for P_{10} invariants and to the construction of the general form of the "free field" Lagrangian density for the Poincaré group. As a consequence, it was shown that P_{10} invariance implied that there were no self-sources in the spin equations (field equations for the compensating fields for the Lorentz sector), and that the space U_4 could be attached to the underlying Minkowski space at a specified center in such a way that the frame bundles, the coframe bundles, and the metric structures could also be attached in a natural way. This attachment process was achieved through implementing a system of antiexact gauge conditions and may thus be applied without loss of generality.

The fundamental question is not that of gauging the Poincaré group, however. We know that the presence of groups of internal symmetries $G(r)$ leads, via the gauge construct, to correct descriptions of elementary matter with strong, weak, and electromagnetic interactions (Salem, 1968; Weinberg, 1974; Marciano and Pagels, 1978). Since these theories are constructed in an underlying Minkowski space, they are manifestly Poincaré covariant. The Poincaré group is thus an external symmetry group for the matter fields as well as the "free field" Lagrangian density for the compensating fields of the internal symmetry group. It thus follows that the inclusion of gravitational interactions through the local action of the Poincaré group necessitates the construction of a gauge theory for the simultaneous local action of both the internal symmetry group and the external symmetry group.

The fact that the Poincaré group is a symmetry group for both the matter fields and the local action of the internal symmetry group indicates that there are two distinct possible modes of approach. The first alternative would be to apply the Poincaré gauge theory starting with the Lagrangian density of the gauge theory for the matter fields and the internal symmetry group. This can be done quite easily by simply replacing the Lagrangian for the matter fields in I and II by the minimal coupling Lagrangian for the matter fields and the local action of the internal symmetry group. The minimal replacement operation for the Poincaré group would then be applied to the compensating 1-forms for the internal symmetry group as well to the matter fields themselves.

The second alternative is to construct a gauge theory for the total symmetry group $P_{10} \times G(r)$, in which case the minimal replacement operation for the total group is applied to the matter fields alone. Now, gauge theory usually starts with the Lagrangian density for the matter fields alone and then proceeds by minimal replacement and minimal coupling constructs to the correct gauge-theoretic field equations. The operable gauge group is discovered, so to speak, by finding the total group of symmetries of the matter Lagrangian density. From this point of view, $P_{10} \times G(r)$ is the total symmetry group for the theory (the group of Noetherian symmetries of the action functional). It would thus appear that the second alternative should prove to be the correct approach. This surmise is directly reinforced by the results established in (Edelen, 1984) where the problem of constructing a gauge theory for a Noetherian symmetry group that acts indiscriminantly on both the matter fields and the base manifold is examined. It is also substantiated by the fact that application of the minimal replacement operator for the Poincaré group to the compensating fields for the internal symmetry group, as would be required in the first alternative, would lead to spin currents that are not $G(r)$ invariant. We therefore restrict our attention in this paper to the problem of gauging the group $P_{10} \times G(r)$ as a total symmetry group for the matter fields.

2. THE TOTAL SYMMETRY GROUP OF THE MATTER FIELDS

The situation of interest is where the Lagrangian 4-form $L(x^i, \Psi^A, \partial_i \Psi^A)\mu$ of the matter fields is jointly invariant under the global action of the Poincaré group, P_{10}, and a semisimple r-parameter group of internal symmetries, $G(r)$. The total symmetry group of the matter fields is thus the direct product group $P_{10} \times G(r)$. We therefore have the problem of constructing a gauge theory for the local action of the total group $P_{10} \times G(r)$.

Let Δ denote the restriction of any group-related quantity to a sufficiently small neighborhood of the identity that second- and higher-order terms in Δ quantities may be neglected. We then have

$$\Delta G(r): \Psi^A \to \Psi^A + \Delta u^b f^A_{bE} \Psi^E, \qquad 1 \leq b \leq r \tag{1}$$

$$\Delta P_{10}: \Psi^A \to \Psi^A + \Delta u^\alpha M^A_{\alpha E} \Psi^E, \qquad 1 \leq \alpha \leq 6 \tag{2}$$

Here, the f's constitute a basis for the Lie algebra of $G(r)$,

$$\mathbf{f}_a \mathbf{f}_b - \mathbf{f}_b \mathbf{f}_a = k^c_{ab} \mathbf{f}_c \tag{3}$$

and the M's constitute a basis for the Lie algebra of the Lorentz sector, $L(4, R)$, of P_{10},

$$\mathbf{M}_\alpha \mathbf{M}_\beta - \mathbf{M}_\beta \mathbf{M}_\alpha = C^\gamma_{\alpha\beta} \mathbf{M}_\gamma \tag{4}$$

where the matter fields constitute the representation space. The reader should note that the translation sector, $T(4)$, of P_{10} does not make a contribution in (2) because the matter fields are assumed to transform under the adjoint representation of P_{10} and thus respond only to the Lorentz sector.

The total group is the direct product of P_{10} and $G(r)$, and hence the action of the infinitesimal transformations of the total group gives

$$\Delta(P_{10}\times G(r)): \Psi^A \to \Psi^A + (\Delta u^\alpha M^A_{\alpha E} + \Delta u^b f^A_{bE})\Psi^E. \tag{5}$$

The results givn in (5) provide explicit evaluations of the deformations (Lie derivatives) of the matter fields that result from the action of $P_{10}\times G(r)$ on the space with local coordinates $\{\Psi^A | 1 \le A \le N\}$. It thus provides direct access to the gauge theory of the total group.

3. MINIMAL REPLACEMENT

We now allow the total group, $P_{10}\times G(r)$, to act locally. Under these circumstances, we may apply the results established in (Edelen, 1984). The direct product structure of the total group allows certain specific simplifications, however. In particular, we may superimpose the individual constructs for P_{10} and $G(r)$. To this end, it is useful to recall the results established in I and II for local action of P_{10}:

$$W^\alpha = W^\alpha_i \, dx^i = \text{compensating 1-forms for } L(4, R) \tag{6}$$

$$\phi^i = \phi^i_k \, dx^k = \text{compensating 1-forms for } T(4) \tag{7}$$

$$B^i = B^i_k \, dx^k = dx^i + W^\alpha l^i_{\alpha_j} x^j + \phi^i = \mathcal{M}(dx^i)$$

$$= \text{distortion 1-forms for local action of } P_{10} \text{ on the underlying Minkowski space } M_4 \text{ (coframe basis for } U_4\text{)}, \tag{8}$$

$$b_i = b^k_i \, \partial_k = \text{frame basis for } U_4 \tag{9}$$

$$b_i \lrcorner B^j = \delta^j_i, \qquad b^k_i B^j_k = \delta^j_i, \qquad \det(B^i_k) = B \ne 0 \tag{10}$$

$$\theta^\alpha = dW^\alpha + C_\beta{}^\alpha{}_\gamma W^\beta \wedge W^\gamma/2 = L(4, R) \text{ curvature 2-forms} \tag{11}$$

$$\Omega^i = d\phi^i + W^\alpha l^i_{\alpha k} \wedge \phi^k = T(4) \text{ curvature 2-forms} \tag{12}$$

$$\Sigma^i = \Omega^i + \theta^\alpha l^i_{\alpha k} x^k = \text{affine Cartan torsion 2-forms} \tag{13}$$

Here, the l's are a basis for the action of the Lie algebra of $L(4, R)$ on the underlying Minkowski space.

The local action of the total group necessitates compensating 1-forms for the $G(r)$ sector. We therefore introduce

$$A^b = A^b_k \, dx^k = \text{compensating 1-forms for } G(r) \tag{14}$$

which give rise to

$$\tilde{\theta}^b = dA^b + k_a{}^b{}_c A^a \wedge A^c/2 = G(r)\text{-curvature 2-forms} \tag{15}$$

Let $\mathcal{M}$ denote the minimal replacement operator. The results established in (Edelen, 1984) and simplified in I show that

$$y_i^A = \mathcal{M}(\partial_i \Psi^A) = b_i^k\{\partial_k \Psi^A + (A_k^b f_{bE}^A + W_k^\alpha M_{\alpha E}^A)\Psi^E\} \tag{16}$$

$$\mathcal{M}(L\mu) = L(x^i, \Psi^A, y_i^A) B\mu \tag{17}$$

Minimal replacement is thus well defined for the local action of the total group $P_{10} \times G(r)$.

There has been some confusion in the literature over whether minimal replacement for P_{10} should or should not be applied to the compensating fields for the internal symmetry group (to Yang–Mills potentials and/or 4-vector potentials of the electromagnetic field). Hehl et al. (1976) conclude that it should not, on the basis that it would lead to spin current 3-forms that break the internal symmetry group. This is the correct answer in view of (16), provided the multiplicative b correction is included. This correction arises from the fact that $d\Psi^A$ is not covariant under Poincaré transformations applied to M_4, while $b_i \lrcorner D\Psi^A$ is a $T(4)$ scalar and hence covariant under Poincaré transformations applied to M_4. Thus, minimal replacement does *not* lead to the transition

$$A_k^b\, dx^k \to A_k^b D x^k = A_k^b B_j^k\, dx^j$$

but rather to

$$A_i^b \to b_i^k A_k^b$$

in the construction of the minimal replacement for $\partial_i \Psi^A$; that is,

$$A_i^b\, dx^i = (b_i^k A_k^b) B^i$$

which is simply the resolution of the A fields on the coframe basis of U_4.

4. MINIMAL COUPLING

The minimal coupling construct of Yang–Mills theory requires us to augment the minimally replaced Lagrangian 4-form of the matter fields by a "free field" Lagrangian 4-form that is invariant under the local action of the total group $P_{10} \times G(r)$. In view of the direct product structure of the total group, the analysis given in Section 5 of II shows that the total Lagrangian density is given by $LB + \Pi B$, where

$$\Pi = \Pi(B_j^i, \theta_{ij}^\alpha, \Sigma_{ij}^k, \tilde{\theta}_{ij}^b) \tag{18}$$

is now a $P_{10}\times G(r)$-invariant scalar. The various possible dependencies of Π on the B's, θ's, and Σ's has been given in II. We therefore have the problem of constructing $P_{10}\times G(r)$-invariant scalars from the B's and the $G(r)$-curvature quantities.

When P_{10} acts locally, the translation sector, $T(4)$, generates all smooth coordinate transformations. Noting that the b's form a frame basis, it follows that

$$\hat{\theta}^b_{ij} = b_i \lrcorner\, b_j \lrcorner\, \tilde{\theta}^b = b_i^m b_j^n \tilde{\theta}^b_{nm} \tag{19}$$

are $T(4)$-invariant scalars. On the other hand, the b's considered as the entries of a row matrix $\underset{\sim}{b}$, transform under the local action of P_{10} by $'\underset{\sim}{b} = \underset{\sim}{b}\underset{\sim}{L}^{-1}$. This shows that there are no P_{10}-invariant scalars that are linear in the $G(r)$-curvature quantities. Noting that

$$\overset{-1}{L}{}^i_m h^{mn} \overset{-1}{L}{}^j_n = h^{ij}$$

where the h's are the components of the inverse of the metric tensor on the underlying Minkowski space, we have the quadratic P_{10}-invariants

$$\hat{\theta}^b_{ij} h^{ik} h^{jm} \hat{\theta}^c_{km}$$

However, (see II-15)

$$b^i_m h^{mn} b^j_n = g^{ij}$$

and hence the quadratic P_{10} invariants assume the simpler form

$$\tilde{\theta}^b_{ij} g^{ik} g^{jm} \tilde{\theta}^c_{km}, \qquad 1 \le b,\, c \le r$$

It now remains to secure $G(r)$ invariance. This, however, is an easy matter since we are in the standard Yang–Mills case with a semisimple internal symmetry group. Thus, if k_{bc} are the components of the Cartan–Killing metric on $G(r)$, we have the quadratic $P_{10}\times G(r)$-invariant "free field" Lagrangian

$$\Pi_G = K\tilde{\theta}^b_{ij} g^{ik} g^{jm} \tilde{\theta}^c_{km} k_{bc} \tag{20}$$

where K is a coupling constant.

This Lagrangian is exactly what we would get for a semisimple internal symmetry group on Minkowski space,

$$\tilde{\theta}^b_{ij} h^{ik} h^{jm} \tilde{\theta}^c_{km} k_{bc}$$

provided we replace h^{ij} by g^{ij}. Thus, the only effect of the factor group P_{10} is this replacement, and the accompanying multiplication by $B=(-g)^{1/2}$ when the corresponding Lagrangian density is constructed. In particular, P_{10} minimal replacement does not apply to the $G(r)$ compensating fields and their derivatives. Further, and of greater importance, invariance theory shows that coupling between the curvature and torsion forms of P_{10} and the curvature forms of $G(r)$ can only occur at cubic and higher order invariants. A restriction to Lagrangians that are at most quadratic in group curvature expressions, as is the usual case, precludes such couplings. Thus, if we use Π_P to denote the quadratic Lagrangian for P_{10} that was obtained in Section 5 of II, the quadratic "free field" Lagrangian density for the total group $P_{10} \times G(r)$ is given by

$$\Pi B = (\Pi_P + \Pi_G) B \tag{21}$$

and we have the superposition principle $\Pi = \Pi_P + \Pi_G$.

5. FIELD EQUATIONS

The field equations for the P_{10} compensating fields will be the same as those reported in I and II with the Lagrangian Π replaced by $\Pi_P + \Pi_G$ in accordance with (21). It is therefore unnecessary to restate them here.

For the matter fields, we introduce the constitutive relations

$$L^i_A = \partial L / \partial y^A_i, \qquad L_A = (\partial L / \partial \Psi^A)|_y \tag{22}$$

Variation of the total Lagrangian, $(L + \Pi)B$, with respect to the matter fields gives the following Euler-Lagrange field equations for the matter fields:

$$\partial_j \{ B b^j_i L^i_A \} - (W^\alpha_j M^E_{\alpha A} + A^b_j f^E_{bA}) \{ B b^j_i L^i_E \} = B L_A \tag{23}$$

If the Poincaré group were restricted so that it acted only globally, the W's would vanish, $B = 1$, and $\underset{\sim}{b}$ would be the identity matrix. In the event, (23) reduce to

$$\partial_j L^j_A - A^b_j f^E_{bA} L^j_E = L_A$$

which are the standard results for an internal symmetry group with compensating 1-forms A^b. If local action of P_{10} is then "switched on", so to speak, we have the two transitions

$$L^j_A \to B b^j_i L^i_A, \qquad A^b_j f^E_{bA} \to A^b_j f^E_{bA} + W^\alpha_j M^E_{\alpha A} \tag{24}$$

The first of these is passive transition that accommodates the transition from Minkowski space to the space U_4 that obtains as a consequence of minimal replacement for the Poincaré group. The second is an active transition whereby "parallelism" corrections obtain as a consequence of the local action of the Lorentz group on the frame and coframe bundles of U_4. We take particular note of the absence of "parallelism" corrections associated with the translation group, $T(4)$, although there are passive contributions from the translation compensating fields in both B and the

b's. This is not altogether an unexpected situation. The $G(r)$ and the $L(4, R)$ sectors are faithfully represented on the matter fields, as shown by (5), while the $T(4)$ sector is not. It is thus reasonable that the $G(r)$ and the $L(4, R)$ sectors give rise to direct, active contributions while the $T(4)$ sector is only passively involved in securing "tensorial" properties under the general coordinate transformations that are generated by the local action of $T(4)$.

Although it should not be heeded too strongly, the absence of active contributions to the matter field equations from the $T(4)$ compensating fields can be taken as further evidence of the reasonableness of imposing the weak constraints of vanishing affine Cartan torsion that were studied in Section 7 of II. Further, we have

$$\mathcal{M}(ds^2) = dS^2 = g_{ij}\, dx^i\, dx^j = B^k h_{km} B^m$$

and

$$D\mathcal{M}(dx^i) = DB^i = \Sigma^i$$

so that

$$D(dS^2) = 2\Sigma^k h_{km} B^m$$

Thus, the weak constraint of vanishing affine Cartan torsion is consistent with metric compatibility demanded by first-order agreement with the Einstein theory.

The field equations for the $G(r)$ compensating fields are most easily handled by introducing the constitutive relations

$$\tilde{G}_b^{ij} = B\partial\Pi/\partial\tilde{\theta}_{ij}^b \tag{25}$$

Variation of the total Lagrangian with respect to the $G(r)$ compensating fields gives the Euler–Lagrange equations

$$2\partial_j \tilde{G}_b^{ji} - 2A_j^a k_a{}^c{}_b \tilde{G}_c^{ji} = Bb^i_m L^m_A f^A_{bE} \Psi^E \tag{26}$$

The left-hand sides of these equations are exactly the same as those that obtain in the standard Yang–Mills gauge theory for an internal semisimple symmetry group. The corresponding terms on the right-hand side, are also recognized as standard after the transition (24) is made. Thus, the only changes in the field equations for the $G(r)$ compensating fields due to the presence of the Poincaré group are the passive ones that render the field equations "tensorial" under the general coordinate transformations generated by the local action of the translation subgroup. These observations, together with the results established in Section 4, show that the field equations for the $G(r)$ compensating fields in the presence of local Poincaré transformations can be written down immediately from knowledge of the corresponding field equations in the absence of local Poincaré transformations.

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